COHOMOLOGY OF LIE TRIPLE SYSTEMS AND LIE ALGEBRAS WITH INVOLUTION(1)

BY

B. HARRIS

A Lie triple system is a subspace of a Lie algebra closed under the ternary composition $[[xy]z]$; equivalently, it may be defined as the subspace of elements mapped into their negatives by an automorphism of order two (an *involution*) in a Lie algebra; or, finally, a Lie triple system may be defined by a set of identities.

Lie triple systems were first noted by E. Cartan in his studies on totally geodesic submanifolds of Lie groups and on symmetric spaces [1] (see also [10]): Lie triple systems are related to totally geodesic submanifolds in the same way that Lie algebras are related to analytic subgroups, and the symmetry in a symmetric space gives rise to the involution in the Lie algebra.

Lie triple systems were studied from the algebraic point of view by Jacobson [6; 7] and Lister [9], the latter giving a complete structure theory including the classification of the simple finite-dimensional systems (in characteristic zero), the Levi decomposition, and (a special case of) the first Whitehead lemma. Simpler axioms were given by Yamaguti [14], who has also studied these and more general systems [15; 16].

In this paper we introduce cohomology groups for Lie triple systems and show that the usual interpretations (in terms of derivations and factor sets) hold for the first and second cohomology groups. In particular we obtain the two Whitehead lemmas (in characteristic zero). Further, these cohomology groups fit into the theory of [2]: they are the cohomology groups of a supplemented associative algebra (very closely related to the universal associative algebra of the Lie triple system).

The groups may briefly be described as follows: if T is any Lie triple system and M a T-module we can construct the universal Lie algebra $L_u(T)$ of T and the "standard extension" M_s which is an $L_u(T)$ module; an involution σ operates on both $L_u(T)$ and M_s such that the elements mapped into their negatives are T and M respectively, and $\sigma([l, n]) = [\sigma(l), \sigma(n)]$ for $l \in L_u(T)$, $n \in M_s$. Now (in a more general situation) if L is any Lie algebra with involution σ and N any L-module also with involution σ, then σ also operates on the cohomology groups $H^n(L, N) = \mathrm{Ext}^n_U(\Phi, N)$ (Φ the base field, U the universal associative algebra of L); we assume further that the characteristic is not 2, so that $H^n(L, N)$ is a direct sum of two subspaces invariant under σ: say $H^n_+(L, N)$, $H^n_-(L, N)$. We show that $H^n_+(L, N) = \mathrm{Ext}^n_K(\Phi, N)$

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where K is the algebra $U+U\sigma$ (with multiplication rules $\sigma\cdot l=\sigma(l)\sigma$ for $l\in L$, $\sigma^2=1$), σ operates as the identity on Φ, and there exists a supplementation of K onto Φ. Coming back to T and the T-module M, we define the nth cohomology group of T with coefficients in M, $H^n(T, M)$, as $H^n_+(L_u(T), M_s)$ $=\mathrm{Ext}^n_K(\Phi, M_s)$. The n-cochains we are considering are thus the multilinear alternating functions f of n variables from $L_u(T)$ to M_s satisfying $f(\sigma(x_1), \cdots, \sigma(x_n))=\sigma f(x_1, \cdots, x_n)$. In particular, we may take the trivial T-module $M=\Phi$: then $M_s=\Phi$ also but with σ acting as -1, and the cochains satisfy $f(\sigma(x_1), \cdots, \sigma(x_n))=-f(x_1, \cdots, x_n)$.

A similar construction of cohomology groups for associative algebras with an anti-automorphism of period two was given in [12]. The analogue for two-sided modules with involution of the cross-product algebra $U+U\sigma$ is used in [5], and the theory of extensions of algebras with a group of operators is given in [13].

1. **Basic definitions.** In this section we recall the basic definitions of Lie triple systems and their enveloping Lie and associative algebras [7; 9] and, following the general idea of [4], define *modules* for Lie triple systems. All algebras, modules, etc. will be assumed to be vector spaces over a field Φ of *characteristic different from two*; all mappings will be assumed linear over Φ.

A Lie triple system T is a vector space over Φ with a ternary composition $[abc]$ which is trilinear and satisfies

$$[aab] = 0, \tag{1.1}$$

$$[abc]+[bca]+[cab] = 0, \tag{1.2}$$

$$[ab[xyz]] = [[abx]yz]+[x[aby]z]+[xy[abz]]. \tag{1.3}$$

Equation (1.3) says that the map $x\to[abx]$ is a Lie triple system derivation.

We will not actually use these identities but use only the fact that they exist.

If L is a Lie algebra, with product $[ab]$, then the ternary composition $[abc]=[[ab]c]$ satisfies the above identities. Conversely, it is shown in [14] or [7] that any Lie triple system T may be considered as a subspace of a Lie algebra L in such a way that $[abc]=[[ab]c]$. The construction is as follows: let L be the vector space direct sum $T\oplus(T\otimes T)^-$, where $(T\otimes T)^-$ is the factor space of $(T\otimes T)^-$ (plain $\otimes$ denoting tensor product over Φ) modulo the subspace of all $\sum a_i\otimes b_i$ such that $\sum[a_ib_ix]=0$ for all x in T. The product $[xy]$ in L is defined by:

$$\begin{aligned}[ab] &= (a\otimes b)^-(a, b \text{ in } T),\\ [(\textstyle\sum[a_ib_i])c] &= \textstyle\sum[a_ib_ic],\\ [c(\textstyle\sum[a_ib_i])] &= -\textstyle\sum[a_ib_ic],\\ [(\textstyle\sum[a_ib_i])(\textstyle\sum[c_id_i])] &= \textstyle\sum[[a_ib_ic_i]d_i]-[[a_ib_id_i]c_i],\end{aligned} \tag{1.4}$$

for $a, b, c, d\in T$.

L is then a Lie algebra, called the *standard* enveloping Lie algebra of T, and denoted by $L_s(T)$ or merely L_s. It is clear that the natural map of T into $L_s(T)$ is 1-1 and $L_s(T)=T\oplus[T, T]$ as vector space.

One can also construct a *universal* enveloping Lie algebra $L_u(T)$, with the property that any homomorphism of T into a Lie algebra extends to a homomorphism of $L_u(T)$, and a universal associative algebra (always with identity) $U(T)$ with the same property relative to homomorphisms of T into associative algebras. $U(T)$ is then also the universal associative algebra of $L_u(T)$, the natural maps of T into $U(T)$ and $L_u(T)$ are 1-1, and $L_u(T)=T\oplus[T, T]$.

From the above remarks we see that a Lie triple system may also be defined as a subspace of a Lie algebra closed under $[[ab]c]$. Most useful for us, however, is the following definition: There exists an automorphism σ of period 2 (i.e., σ^2 = identity map) in a Lie algebra L such that $T=\{l\in L\mid\sigma(l)=-l\}$. In $L_u(T)=T\oplus[T, T]$ the automorphism is: $\sigma(t)=-t$, $\sigma([t_1t_2])=[t_1t_2]$ for t, t_i in T. σ will be called the *involution.*

A vector space M is called a *module* for the Lie triple system T if the vector space direct sum $E=T\oplus M$ is itself a Lie triple system in such a way that: (1) T is a subsystem, (2) $[abc]$ lies in M if any *one* of a, b, c is in M, (3) $[abc]=0$ if any *two* of a, b, c are in M. Equivalently, a module M is a vector space together with a map of $M\otimes T\otimes T$ into M satisfying the identities (1.1)–(1.3) (and condition (3) above). $E=T\oplus M$ will be called the semi-direct sum, or split null extension, or inessential extension, of T by M.

We may also describe a Lie triple system module M in terms of a Lie algebra module and an involution. For this purpose we define the standard extension $M_s=N_s(M)$, which will be a Lie algebra module over $L_u(T)$: let $N=N_s(M)$ be the vector space direct sum $M\oplus(M\otimes T)^-$ where $(M\otimes T)^-$ is the quotient space $M\otimes T$ modulo the subspace $\{\sum m_i\otimes t_i\mid\sum[m_i, t_i, t]=0$ for all $t\in T\}$. The rules giving the structure of $N_s(M)$ and $L_u(T)$ module are the obvious analogues of (1.4); to show these rules are well defined we proceed as follows: first form the semi-direct sum $E=T\oplus M$ and its standard Lie algebra (see [7]) $L_s(E)=E\oplus(E\otimes E)^-$. The natural injection of $M\otimes T$ into $E\otimes E$ gives an injection of $(M\otimes T)^-$ into $(E\otimes E)^-$ (the conditions that $\sum(m_i\otimes t_i)^-$ be zero in $(M\otimes T)^-$ or $(E\otimes E)^-$ are the same, namely $\sum[m_i, t_i, t]=0$ in M for all $t\in T$, since $\sum[m_i, t_i, m]=0$ automatically for $m\in M$). Thus $N_s(M)$ may be considered as a subspace of $L_s(E)$ and is in fact an ideal in $L_s(E)$. Since $L_s(E)$ is a Lie algebra containing the Lie triple system T, the identity map $T\to T\subset L_s(E)$ may be extended to a homomorphism of $L_u(T)$ into $L_s(E)$: $N_s(M)$ being an ideal in $L_s(E)$, then becomes also an $L_u(T)$ module.

Now define the involution σ in $L_s(E)$ by $\sigma(x)=-x$ for $x\in E$, $\sigma(x)=x$ for $x\in[E, E]$. Then σ maps $N_s(M)$ into itself and $\sigma(m)=-m$, $\sigma([t, m])=[t, m]$ for $m\in M$, $t\in T$. It is then clear that $\sigma([l, n])=[\sigma(l), \sigma(n)]$ for $l\in L_u(T)$, $n\in N_s(T)$ and that $M=\{n\in N_s(T)\mid\sigma(n)=-n\}$.

The module $N_s(T)$ of $L_u(T)$ may also be considered as a module over $U(T)$, which we will write as a *left* module by: $x \cdot n = [x, n] = -[n, x]$ for $x \in L_u(T) \subset U(T)$. The automorphism σ of $L_u(T)$ induces an automorphism, also denoted by σ, of period 2 in $U(T)$, and

$$(1.5) \qquad \sigma(u \cdot n) = \sigma(u) \cdot \sigma(n), \qquad \text{for } u \in U(T), n \in N_s(T).$$

In general, if N is a left module for an associative algebra U, or a Lie algebra L, and there exists a 1-1 map σ of N onto itself, and an involution, also denoted by σ, in U, or in L, satisfying (1.5), N is called a *module with involution*. Equivalently, a left U-module with involution is a left module over an algebra K constructed as follows: as vector space $K = U \oplus U$, and the multiplication is

$$(a \oplus b)(c \oplus d) = (ac + b\sigma(d)) \oplus (ad + b\sigma(c)).$$

We shall write $a + b\sigma$ for $a \oplus b$, so that the multiplication rules in K are: $\sigma a = a\sigma(a)$, $\sigma^2 = 1$; we shall write $U + U\sigma$ for K. For $n \in N$, we shall write $\sigma \cdot n$ for $\sigma(n)$: thus if N is a left U-module with involution, this rule makes N a left K-module, while if N is a left K-module it is also a left U-module with the involution σ.

If T is a Lie triple system, M a module for T, then in general there are many Lie algebra modules with involution N for $L_u(T)$ such that $N = M \oplus [T, M]$ (vector space direct sum) and $M = \{n \in N | \sigma(n) = -n\}$. Among these the standard extension $N_s(M)$ may be characterized by the fact that the identity map $M \to M$ extends to a homomorphism of N *onto* $N_s(M)$ (for the map $m + \sum [t_i, m_i] \to m + \sum (t_i \otimes m_i)^-$ of N into $N_s(M)$ is well defined, and clearly a homomorphism onto).

In the same way the standard extension $L_s(T)$ and the universal extension $L_u(T)$ may be characterized among all enveloping Lie algebras L of T that satisfy $L = T \oplus [T, T]$: There is a (unique) homomorphism of $L_u(T)$ onto L and one of L onto $L_s(T)$ which are the identity on T. In particular there is a (unique) homomorphism of $L_u(T)$ on $L_s(T)$ which is the identity on T: its kernel is the intersection of the center of $L_u(T)$ with $[T, T]$. Thus if $L_u(T)$ has no center, then $L_u(T)$ is isomorphic to $L_s(T)$.

These constructions have the following functorial properties:

1. If M, M' are modules for the Lie triple system T, and ϕ is a module homomorphism of M into M', then ϕ induces a homomorphism ϕ_s of the $L_u(T)$ module $M_s = N_s(M)$ into M'_s. If ϕ is 1-1 or onto, then ϕ_s is 1-1 or onto, respectively.

2. If T, T' are Lie triple systems and ϕ a homomorphism of T into T', then ϕ induces a homomorphism ϕ_u of $L_u(T)$ into $L_u(T')$, which is onto if ϕ is onto. Further, if ϕ is onto, it induces a homomorphism ϕ_s of $L_s(T)$ onto $L_s(T')$.

The above homomorphisms all commute with the involution σ.

2. **Cohomology of Lie algebras with involution.** In this section we consider

Lie algebras L with involution σ, and L-modules N with involution, and study the effect of σ on the cohomology groups of L with coefficients in N. In the next section we will apply these results to $L=L_u(T)$, T a Lie triple system, and $N=M_s=N_s(M)$, M a T-module.

The cohomology groups $H^n(L, N)$ of L with coefficients in N are defined [2, Chapter 13] as $\mathrm{Ext}^n_U(\Phi, N)$ where $U=U(L)$ is the universal enveloping associative algebra of L, the structure of the base field Φ as U-module being given by a supplementation $\epsilon: U\rightarrow\Phi$. We also have present the involution σ so that we can form $K=U+U\sigma$: we shall show that K has some of the important properties of U, and will study the effect of replacing U to K.

The properties of U we shall consider are [2, Chapters 10, 13]:

1. The supplementarion ϵ: $U\rightarrow\Phi$.

2. A homomorphism E of U into $U^e=U\otimes U^*$ (U^* being the algebra anti-isomorphic to U under the map $u\rightarrow u^*$) such that

$$\begin{array}{ccc} U & \xrightarrow{\epsilon} & \Phi \\ E\downarrow & & \downarrow\eta \\ U^e & \xrightarrow[\rho]{} & U \end{array}$$

is a commutative diagram (η denoting the map $\alpha\rightarrow\alpha 1$, ρ the map $a\otimes b^*\rightarrow ab$), i.e. $E(I)\subset J$, I being the kernel of ϵ and J that of ρ.

3. $J=U^e\cdot E(I)$ and U^e is a projective right U module (U^e being a right U-module under the map $(a\otimes b^*)\cdot c=(a\otimes b^*)E(c)$ for $c\in U$) so that [2, Theorem 10.6.1], if we make a left U^e-module N into a left U-module ${}_EN$ by the map E, then $\mathrm{Ext}^n_U(\Phi, {}_EN)$ is isomorphic to $\mathrm{Ext}^n_{U^e}(U, N)$.

4. There is an "antipodism" (isomorphism) ω of U with U^* so that $D=\omega E$: $U\rightarrow^E U\otimes U^*\rightarrow^{1\otimes\omega} U\otimes U$ is a homomorphism (actually an isomorphism), called a diagonal map.

Since U is a subalgebra of K, and U^e of $K^e=K\otimes K^*$, we shall use the same letters as above for the maps involving K. We define the supplementation ϵ of K onto Φ by $\epsilon(\sigma)=1$, and ϵ as before on U, i.e., $\epsilon(a+b\sigma)=\epsilon(a)+\epsilon(b)$. The kernel I_0 then contains the kernel I of the restriction of ϵ to U: in fact $I_0=I+I\sigma+\Phi(1-\sigma)$. Since $\sigma(I)=I$ in U and $U=\Phi 1+I,_0$ it is clear that ϵ is a homomorphism of K onto Φ.

The map E of U into U^e will be extended to a map $K\rightarrow K^e$ by setting $E(\sigma)=\sigma\otimes\sigma^*$: i.e., we let $E(a+b\sigma)=E(a)+E(b)(\sigma\otimes\sigma^*)$. For $x\in L\subset U$, $E(x)=x\otimes 1-1\otimes x^*$. It is easy to check that $E(\sigma)E(x)=E(\sigma x)$ for $x\in L$, and since L and 1 generate U, $E(\sigma)E(a)=E(\sigma a)$ for all $a\in U$. Finally, E is a homomorphism. E is 1-1 since the homomorphism $1\otimes\epsilon^*$: $K\otimes K^*\rightarrow K$ by $(1\otimes\epsilon^*)(a\otimes b^*)=a\epsilon(b)$ satisfies $(1\otimes\epsilon^*)E(x)=x$, $(1\otimes\epsilon^*)E(\sigma)=\sigma$, $(x\in L)$ and so $(1\otimes\epsilon^*)E(k)=k$ for $k\in K$.

The kernel J_0 of the map $K^e\rightarrow K$ is the left ideal in K^e generated by the elements $(k\otimes 1-1\otimes k^*)$ and so J_0 contains J, the kernel of the map $U^e\to U$. We show first that $J_0=K^e\cdot E(I_0)$: $I_0=I+I\sigma+\Phi(1-\sigma)=I+\sigma I+\Phi(1-\sigma)$; it is known that $E(I)\subset J\subset J_0$, thus $E(\sigma I)=E(\sigma)E(I)\subset J_0$ also, since J_0 is a left ideal, and finally $E(1-\sigma)=1\otimes 1^*-\sigma\otimes\sigma^*=(\sigma\otimes 1)(\sigma\otimes 1^*-1\otimes\sigma^*)\in J$. Thus $E(I_0)\subset J_0$ and $J_0\supset K^e\cdot E(I_0)$. Conversely, we have to show $k\otimes 1-1\otimes k^*\in K^e\cdot E(I_0)$ for $k\in K$. Let $k=a+b\sigma$, $k\otimes 1-1\otimes k^*=(a\otimes 1-1\otimes a^*)+(b\sigma\otimes 1-1\otimes(b\sigma)^*)$. $(a\otimes 1-1\otimes a^*)\in J=U^e\cdot E(I)\subset K^e\cdot E(I_0)$. $(b\sigma\otimes 1-1\otimes(b\sigma)^*)=(b\sigma\otimes 1-1\otimes\sigma^*b^*)=(b\otimes 1)(\sigma\otimes 1-1\otimes\sigma^*)+(1\otimes\sigma^*)(b\otimes 1-1\otimes b^*)$. Since $(\sigma\otimes 1-1\otimes\sigma^*)=(\sigma\otimes 1)E(1-\sigma)\in K^eE(I_0)$, $k\otimes 1-1\otimes k^*\in K^e\cdot E(I_0)$ also. Next, K^e_E is right K-projective (K^e_E denoting K^e as right K-module under the map E): to show this we first define the isomorphism ω of K and K^* by $\omega(\sigma)=\sigma^*$ and, on U, (if $x_i\in L$), $\omega(x_1\cdots x_p)=(-1)^p(x_p^*\cdots x_1^*)$ so that $\omega(a+b\sigma)=\omega(a)+\omega(b)\omega(\sigma)$, so that $D=(1\otimes\omega)E$ is an isomorphism of K into $K\otimes K$. $((1\otimes\omega)(a\otimes b^*)=a\otimes\omega(b^*))$. Then K^e_E as right K-module is isomorphic to $(K\otimes K)_D$ and $D(x)=x\otimes 1+1\otimes x$ for $x\in L$, $D(\sigma)=\sigma\otimes\sigma$. $(U\otimes U)_D$ is a free right U-module [2, Chapter 13, Proposition 4.1] with a basis consisting of monomials $m_i=(x_1\otimes 1)\cdots(x_n\otimes 1)$, $\{x_i\}$ a basis for L (since $U\otimes U=U(L\oplus L)$ and the subalgebra $\{x\otimes 1+1\otimes x\}$ of $L\oplus L$ has a basis of the form $\{x_i\otimes 1+1\otimes x_i\}$, while these elements together with the $\{x_i\otimes 1\}$ form a basis for $L\oplus L$). In fact we may also assume that $x_i^\sigma=\sigma(x_i)=\pm x_i$, and so, in $K\otimes K$, $m_i(\sigma\otimes\sigma)=\pm(\sigma\otimes\sigma)m_i$.

Since $K=U+\sigma U$, $K\otimes K$ is the vector space direct sum $U\otimes U\oplus(\sigma\otimes\sigma)U\otimes U\oplus(\sigma\otimes 1)U\otimes U\oplus(1\otimes\sigma)U\otimes U$. Since $U\otimes U=\sum\oplus m_i\cdot U$ as right U-module, $U\otimes U+(\sigma\otimes\sigma)U\otimes U=\sum\oplus m_i\cdot K$ is a free right K-module with generators m_i: first, the m_i are generators since $(\sigma\otimes\sigma)m_i=\pm m_i(\sigma\otimes\sigma)=\pm m_i\cdot\sigma=\pm m_iD(\sigma)$, and if $\sum m_i\cdot k_i=\sum m_iD(k_i)=0$ for $k_i\in K$, $k_i=a_i+\sigma b_i$ then $0=\sum m_iD(a_i)\pm(\sigma\otimes\sigma)m_iD(b_i)\in U\otimes U\oplus(\sigma\otimes\sigma)(U\otimes U)$ and so $\sum m_iD(a_i)=0=\sum m_iD(b_i)$. But since the m_i are a basis for $(U\otimes U)_D$, $D(a_i)=D(b_i)=D(k_i)=0$. In the same way, the elements $(\sigma\otimes 1)m_i$ are a basis for $(\sigma\otimes 1)U\otimes U\oplus(1\otimes\sigma)U\otimes U$. Thus $(K\otimes K)_D$ is right K-free.

Thus, by [2, Chapter 10, Theorem 10.6.1] we have

THEOREM 2.1. *If A is any left K^e-module, so that ${}_EA$ is a left K-module, then* $\operatorname{Ext}^n_K(\Phi, {}_EA)$ *is isomorphic to* $\operatorname{Ext}^n_{K^e}(K, A)$.

COROLLARY 2.1. *Let M, N be left K-modules, then* $\operatorname{Ext}^n_K(M, N)$ *is isomorphic to* $\operatorname{Ext}^n_K(\Phi, \operatorname{Hom}_\Phi(M, N))$.

[$\operatorname{Hom}_\Phi(M, N)$ is a left K-module under $(\sigma\cdot h)(m)=\sigma h(\sigma(m))$ and $(x\cdot h)(m)=x\cdot h(m)-h(x\cdot m)$ for $x\in L$.]

Proof. The above definition of $H=\operatorname{Hom}_\Phi(M, N)$ as left K-module is the same as starting with H as left K^e module in the usual way: $(k\cdot h)(m)=k\cdot h(m)$, $(h\cdot k)(m)=h(k\cdot m)$ and then making it into the left K-module ${}_EH$. Then, $\operatorname{Ext}^n_K(\Phi, {}_EH)$ is isomorphic to $\operatorname{Ext}^n_{K^e}(K, H)$ and, by [2, Chapter 9, Corollary 4.4], $\operatorname{Ext}^n_{K^e}(K, \operatorname{Hom}_\Phi(M, N))$ is isomorphic to $\operatorname{Ext}^n_K(M, N)$.

Next, we note that K is projective, even free, as left (or right) U-module, since $K=U+U\sigma=U+\sigma U$. Therefore, for any left U-module A, $\mathrm{Tor}_n^U(K, A)=0$ for $n>0$, and so, by [2, p. 118] we have the isomorphism

$$\mathrm{Ext}_K^n(K\otimes_U A, C)\simeq \mathrm{Ext}_U^n(A, C), \tag{2.1}$$

for any left K-module C. In particular, let $A=\Phi$ as left U-module: then $K\otimes_U\Phi=U\otimes_U\Phi+\sigma U\otimes_U\Phi=1\otimes_U\Phi+\sigma\otimes_U\Phi$. At this point we employ the elements $e=(1+\sigma)/2$, $f=(1-\sigma)/2$ which are orthogonal idempotents in K with sum 1; then $K\otimes_U\Phi=e\otimes_U\Phi\oplus f\otimes_U\Phi$ (direct sum as left K-modules). $e\otimes_U\Phi$ is isomorphic to Φ as left K-module as defined before (i.e., with σ acting as the identity) under the map $e\otimes_U\alpha\to\alpha$, while $\sigma(f\otimes_U\alpha)=-f\otimes_U\alpha$ (since $\sigma f=-f$) so that $f\otimes_U\Phi$, which will be denoted by Φ^- with $f\otimes_U\alpha$ denoted by α, is isomorphic to Φ as left U-module but σ acts as -1 on it. Then (2.1) becomes

$$\mathrm{Ext}_U^n(\Phi, C)\simeq \mathrm{Ext}_K^n(\Phi, C)\oplus \mathrm{Ext}_K^n(\Phi^-, C)\qquad \text{for any left } K\text{-module } C. \tag{2.1$'$}$$

We now proceed to construct a specific resolution of Φ as left K-module. We start with the usual resolution of Φ as U-module.

$$\cdots\to U\otimes\Lambda_n\xrightarrow{d_n} U\otimes\Lambda_{n-1}\to\cdots\to U\xrightarrow{\epsilon}\Phi\to 0 \tag{2.2}$$

where Λ is the exterior algebra on the vector space L, Λ_n the subspace of homogeneous elements of degree n of Λ, and

$$\begin{aligned} d_n(a\otimes x_1\wedge\cdots\wedge x_n) &= \sum_{i=1}^{n}(-1)^{i+1}ax_i\otimes x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_n \\ &\quad+\sum_{i<j}(-1)^{i+j}a\otimes[x_ix_j]\wedge x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n. \end{aligned} \tag{2.3}$$

Following the reasoning [2, Chapter 6, §4] leading to (2.1) and (2.1′), we form the left K-modules $Y_n=K\otimes_U(U\otimes\Lambda_n)\simeq K\otimes\Lambda_n$ and $K\otimes_U\Phi\simeq\Phi\oplus\Phi^-$. Since $Y_n=K\otimes_U X_n$ and the X_n are a U-projective resolution of Φ with the differentiation d_n of (2.3), if we extend d_n to the Y_n by $d_n(k\otimes_U x_n)=k\otimes_U(d_nx_n)$, the Y_n with this differentiation will be a K-projective resolution of $K\otimes_U\Phi$ (since K is U-projective). Writing Y_n as $K\otimes\Lambda_n$ we see that d_n is given by the formula (2.3) again (but a, in (2.3), being now an element of K). The map $K=Y_0\to K\otimes_U\Phi$ is given by $\sigma\to\sigma\otimes_U 1$, $a\to\epsilon(a)$ for $a\in U$.

The direct sum decomposition $K\otimes_U\Phi=\Phi\oplus\Phi^-$ will now be extended to a direct sum decomposition (as K-module) $K\otimes_U X_n=Y_n=Y_n^+\oplus Y_n^-$ such that the Y_n^+ (with differentiation induced by d_n) form a K-projective resolution of Φ, and the Y_n^- of Φ^-. We set

$$\begin{aligned} Y_n^+ &= \{a\otimes\lambda_n+a\sigma\otimes\lambda_n^\sigma \mid a\in U, \lambda_n\in\Lambda_n\} \\ &= (Ke\otimes\Lambda_n^+)\oplus(Kf\otimes\Lambda_n^-) \end{aligned} \tag{2.4}$$

where $\lambda \to \lambda^\sigma$ is the automorphism of Λ induced by the automorphism σ of L, and $\Lambda_n{}^+ = \{\lambda_n \in \Lambda_n | \lambda_n^\sigma = \lambda_n\}$, $\Lambda_n{}^- = \{\lambda_n \in \Lambda_n | \lambda_n^\sigma = -\lambda_n\}$ (recall that $e=(1+\sigma)/2$, $f=(1-\sigma)/2$). Similarly,

$$
\begin{aligned}
Y_n{}^- &= \{a \otimes \lambda_n - a\sigma \otimes \lambda_n^\sigma \mid a \in U, \lambda_n \in \Lambda_n\} \\
&= (Ke \otimes \Lambda_n{}^-) \oplus (Kf \otimes \Lambda_n{}^+).
\end{aligned}
\tag{2.4'}
$$

To show that $d_n(Y_n{}^+) \subset Y_{(n-1)}{}^+$ and $d(Y_n{}^-) \subset Y_{(n-1)}{}^-$ we note that, if we write $(a \otimes \lambda_n)\cdot\sigma$ for $a\sigma \otimes \lambda_n^\sigma (a \in U, \lambda_n \in \Lambda_n)$ then d_n satisfies

$$
d_n[(a \otimes \lambda_n)\cdot\sigma] = [d_n(a \otimes \lambda_n)]\cdot\sigma,
\tag{2.5}
$$

therefore

$$
\begin{aligned}
d_n(a \otimes \lambda_n + a\sigma \otimes \lambda_n^\sigma) &= d_n(a \otimes \lambda_n + (a \otimes \lambda_n)\cdot\sigma) \\
&= d_n(a \otimes \lambda_n) + (d_n(a \otimes \lambda_n))\cdot\sigma \in Y_{(n-1)}{}^+.
\end{aligned}
$$

The map $Y_0 = K$ onto $K \otimes_U \Phi = \Phi \oplus \Phi^- = e \otimes_U \Phi \oplus f \otimes_U \Phi$ is given by $\sigma \to \sigma \otimes_U 1$, and $a \to \epsilon(a)\cdot 1$ for $a \in U$. Thus $e=(1-\sigma)/2 \to e \otimes_U 1 \in \Phi$, and $f \to f \otimes_U 1$ in Φ^-. This map therefore sends $Y_0{}^+$ onto Φ, $Y_0{}^-$ onto Φ^-, and its kernel in $Y_0{}^+$ is $d_1(Y_1{}^+)$ and similarly for $Y_0{}^-$.

Finally, the $Y_n{}^+$ furnish a projective resolution of Φ as left K-module, and the $Y_n{}^-$ of Φ^-.

Let now N be a left K-module, and let

$$
C^n = \mathrm{Hom}_K(Y_n, N), \qquad C_+^n = \mathrm{Hom}_K(Y_{n^+}, N), \qquad C_-^n = \mathrm{Hom}_K(Y_{n^-}, N)
$$

so that $C^n = C_+^n \oplus C_-^n$. The map d_n then induces a map $\delta^n: C^n \to C^{n+1}$, $C_+^n \to C_+^{n+1}$, $C_-^n \to C_-^{n+1}$. It is clear that C^n, for $n \geqq 1$, may be identified with the vector space of all multilinear alternating functions of n variables in L with values in N, while C_+^n consists of those functions f which satisfy the additional condition

$$
f(x_1^\sigma, \cdots, x_n^\sigma) = \sigma(f(x_1, \cdots, x_n))
\tag{2.6}
$$

and similarly the $f \in C_-^n$ satisfy $f(x_1^\sigma, \cdots, x_n^\sigma) = -\sigma(f(x_1, \cdots, x_n))$.

We let Z^n, Z_+^n, Z_-^n be the kernels of δ^n in C^n, C_+^n, C_-^n respectively, also B^n, B_+^n, B_-^n the images of δ^{n-1}, and H^n, H_+^n, H_-^n be the quotient spaces Z^n/B^n, Z_+^n/B_+^n, Z_-^n/B_-^n respectively. Thus $H^n = H_+^n \oplus H_-^n$. H_+^n, which we shall denote also by $H_+^n(L, N)$, is thus obtained by taking those functions f on n variables which are cocycles for the usual coboundary operator and satisfy in addition (2.6), modulo the coboundaries of functions satisfying (2.6) with $n-1$ for n (or, equivalently, modulo those coboundaries which satisfy (2.6)).

Next we consider interpretations of the groups $H_+^n(L, N)$ for $n = 0$, 1 and 2.

$H_+^0 = H_+^0(L, N) = \mathrm{Hom}_K(Ke, N)/\mathrm{Hom}_K(\Phi, N)$ consists of the subspace of $N_+ = eN$ of elements n such that $[x, n] = 0$ for all $x \in L$, i.e. the submodule of *invariants* of N_+.

$H^1_+ = H^1_+(L, N)$ consists of those derivations f of L into N such that $f(\sigma(x)) = \sigma(f(x))$, modulo inner derivations of the form $f(x) = [x, n]$ with $n \in N_+ = eN$.

Another interpretation of H^1_+ is in terms of extensions of K-modules modulo inessential extensions: if N, N' are K-modules and E an extension of N by N', i.e., $0 \to N' \to^i E \to^p N \to 0$ is exact, then the group (as defined in [2, Chapter 14]) of such extensions E modulo the inessential extensions, is isomorphic to $\mathrm{Ext}^1_K(N, N')$, which, by Corollary 2.2, is isomorphic to $H^1_+(L, \mathrm{Hom}_\Phi(N, N')) = \mathrm{Ext}^1_K(\Phi, \mathrm{Hom}_\Phi(N, N'))$.

The group $H^2_+(L, N)$ is connected with extensions of Lie algebras with involution, i.e., extensions $0 \to N \to^i E \to^p L \to 0$ such that N, E, L are all Lie algebras with involution σ (N being an abelian ideal in E) and i, p commute with σ. Such an extension E yields a 2-cocycle $g \in Z^2_+(L, N)$: choose a linear map q of L into E such that $q\sigma = \sigma q$ and $pq(l) = l$ for $l \in L$ (a "linear inverse" to p), and let $g(x, y) = [q(x), q(y)] - q([x, y])$ for $x, y \in L$. Then $g(x^\sigma, y^\sigma) = \sigma g(x, y)$ and $\delta g = 0$. Any other choice of a map q satisfying these conditions would merely alter g by the addition of an element of $B^2_+(L, N)$.

Conversely, given $g \in Z^2_+(L, N)$ one can construct an extension E as the vector space direct sum $E = N \oplus L$, with multiplication

$$[(n_1 \oplus x), (n_2 \oplus y)] = ([n_1, y] - [n_2, x] + g(x, y)) \oplus [x, y]$$

and involution $\sigma(n \oplus x) = \sigma(n) \oplus \sigma(x)$. Then E is a Lie algebra with involution, and the maps $i: n \to n \oplus 0$, $p: (n \oplus x) \to x$ of $N \to E$, $E \to L$, commute with σ.

One can then verify that the correspondence $g \leftrightarrow E$ gives a 1-1 correspondence between $H^2_+(L, N)$ and the equivalence classes of extensions E of L by N (as Lie algebras with involution); the zero element of $H^2_+(L, N)$ corresponds to the class of inessential extensions E, i.e., those for which there is a Lie algebra homomorphism q of L into N such that $q\sigma = \sigma q$ and $pq(l) = l$ for $l \in L$.

3. **Cohomology of Lie triple systems.** Let T be a Lie triple system, M a T-module. We define the nth cohomology group of T with coefficients in M, $H^n(T, M)$ as the group $H^n_+(L_u(T), M_s)$ defined in the previous section, where $L_u(T)$ is the *universal* enveloping Lie algebra of T and M_s is the *standard* extension of M, and the involution σ acts on $L_u(T)$ and on M_s as explained in §1. Thus $H^n_+(T, M) = \mathrm{Ext}^n_K(\Phi, M_s)$ and is a direct summand of the ordinary nth cohomology group $H^n(L_u(T), M_s)$.

We start by examining the 0th, 1st, and 2nd cohomology groups for arbitrary T and M, making no assumption on the dimension of T and M as vector spaces over the field Φ or on the characteristic of Φ (except the standing assumption that the characteristic is not two); at the end of the section we specialize to finite-dimensional semi-simple T and Φ of characteristic zero, and obtain the analogues of the two Whitehead lemmas.

As shown in §2, $H^0_+(L_u(T), M_s)$ consists of the invariants of $(M_s)_+ = eM_s = [T, M]$: by the definition of M_s, however, the only such invariant is zero. Thus $H^0(T, M) = (0)$ for all T, M.

We define a *derivation* D of T into M as a linear transformation satisfying

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]. \tag{3.1}$$

($[a, b, c]$ denotes $[[ab]c]=[c[ba]]$ whenever it is permissible to use the operation $[ab]$.) Equivalently, D is a derivation of T into M if and only if the map $h: x \to x \oplus D(x)$ of T into the semi-direct sum $E = T \oplus M$ is an isomorphism into E.

PROPOSITION 3.1. *Every Lie triple system derivation D of T into M can be extended uniquely to a Lie algebra derivation D_0 of $L_u(T)$ into M_s such that $D_0\sigma=\sigma D_0$; conversely, every Lie algebra derivation D_0 of $L_u(T)$ into M_s satisfying $D_0\sigma=\sigma D_0$ induces a Lie triple system derivation of T into M. D_0 is inner if and only if D has the form*

$$D(t) = \sum_i [t, [t_i, m_i]], \qquad t_i \in T, m_i \in M. \tag{3.2}$$

Proof. Let $G = L_u(T) \oplus M_s$ (semi-direct sum) and $G_- = T \oplus M$, $G_+ = [T, T] \oplus [T, M]$ so that $G = G_- \oplus G_+$. Let D be a derivation of T into M, $h: t \to t \oplus D(t)$ the corresponding homomorphism of T into $E = T \oplus M$. Since $L_u(T)$ is the universal Lie algebra of T and since h is a homomorphism of T into the Lie algebra $G = E \oplus [T, E]$, h can be extended uniquely to a homomorphism h_0 of $L_u(T)$ into G.

Since $h(t) - t \in M$ for $t \in T$, and since h_0 is a homomorphism of Lie algebras, $h_0([t_1, t_2]) - [t_1, t_2] \in [T, M]$ for $t_i \in T$, and so $h_0(l) - l \in M \oplus [T, M] = M_s$ for $l \in L_u(T)$. Thus $h_0(l) = l \oplus D_0(l) \in L_u(T) \oplus M_s$ and D_0 is a derivation of $L_u(T)$ into M_s which agrees with D on T. Since $D_0(T) \subseteq M$, $D_0([T, T]) \subseteq [T, M]$, and $\sigma D_0 = D_0\sigma$. D_0 is uniquely determined by D since T generates $L_u(T)$. Conversely, it is clear that any D_0 satisfying $D_0\sigma=\sigma D_0$ maps T into M and so induces a Lie triple system derivation of T into M.

Finally, let D_0 be inner: $D_0(l) = [l, n]$ where $n = n_- \oplus n_+ \in M \oplus [T, M]$. Taking $l = t \in T$ and using the fact that $D_0\sigma = \sigma D_0$, we find that $[t, n_-] = 0$ and $D_0(t) = [t, n_+]$ (so that $D_0(l) = [l, n_+]$ for all $l \in L_u(T)$ also). But n_+, being in $[T, M]$, has the form $\sum_i [t_i, m_i]$, and so $D(t) = D_0(t)$ satisfies (3.2). The converse is clear. Thus the proof is complete.

A Lie triple system derivation is called *inner* if it satisfies (3.2). Thus Proposition 3.1, together with the considerations of §2, yields

THEOREM 3.1. *The group $H^1(T, M)$ is isomorphic to the group of derivations of T into M modulo the subgroup of inner derivations.*

Next we consider algebra extensions and the second cohomology group.

Let E, T be Lie triple systems, p a homomorphism of E onto T with kernel M: thus $0 \to M \to^i E \to^p T \to 0$ is exact, i denoting the injection. We shall always assume that, if $x, y, z \in E$, $[x, y, z] = 0$ if any two of x, y, z are in M: then M is a T-module in the natural way. All such extensions E of T by M

(M being a T-module) may be constructed by means of factor sets $f(x, y, z)$, which are alternating tri-linear maps of $T\times T\times T$ into M satisfying certain identities: as vector space $E=M\oplus T$, and the multiplication in E is given by

$$(3.3)\qquad [(0\oplus t_1), (0\oplus t_2), (0\oplus t_3)] = f(t_1, t_2, t_3)\oplus [t_1, t_2, t_3]$$

together with the T-module structure of M. Thus f has to satisfy certain identities obtained from the multilinear identities (1.1)–(1.3); further, each of these identities consists of sums of monomials which are of first degree in f, so that the factor sets form a vector space over the base field. We shall show that the group (actually vector space) of factor sets modulo the subgroup (subspace) of trivial factor sets is isomorphic to the second cohomology group $H^2(T, M)$: the trivial factor sets are defined to be those associated with inessential extensions E, i.e., those extensions for which there exists a Lie triple system homomorphism q of T into E such that $pq(t)=t$ for $t\in T$; equivalently, a factor set f is trivial if there is a linear map h of T into M such that

$$(3.4)\quad f(t_1, t_2, t_3) = h([t_1, t_2, t_3]) - [h(t_1), t_2, t_3] + [h(t_2), t_1, t_3] - [t_1, t_2, h(t_3)]$$

(see [9]).

THEOREM 3.2. *The group $H^2(T, M)$ is isomorphic to the group of factor sets of T into M modulo the subgroup of trivial factor sets.*

Proof. We shall associate extensions of Lie triple systems to extensions of Lie algebras with involution, and then use the correspondence between the latter and elements of $H^2_+(L_u(T), M_s)=H^2(T, M)$ discussed in §2. The main step is:

LEMMA. *Let $0\to M\to^{i} E\to^{p} T\to 0$ be an extension of T by M. Then there exists an extension of Lie algebras with involution $0\to M_s\to^{i_0} G\to^{p_0} L_u(T)\to 0$ of $L_u(T)$ by M_s such that the following diagram is commutative (j denoting injections):*

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{i} & E & \xrightarrow{p} & T & \to & 0 \\ & & \downarrow j & & \downarrow j & & \downarrow j & & \\ 0 & \to & M_s & \to & G & \to & L_u(T) & \to & 0. \end{array}$$

Proof. The homomorphism p of E onto T induces a homomorphism p' of $L_u(E)$ onto $L_u(T)$ which commutes with σ. We show first that the kernel of p' is the ideal $I=M+[E, M]$ in $L_u(E)$ generated by M. We will then define G as the quotient of $L_u(T)$ by another ideal $J\subset I$ such that I/J is isomorphic to M_s.

It is clear that $I=M+[E, M]\subset$kernel of p' since $M\subset$kernel of p'. On the other hand, the natural homomorphism r of $L_u(E)$ onto $L_u(E)/I$ induces a homomorphism of E into $L_u(E)/I$ and maps M into zero, thus r can be factored through $L_u(T)$,

$$\begin{array}{ccc} & L_u(E) & \\ r\swarrow & & \searrow p' \\ L_u(E)/I & \overset{q}{\longleftarrow} & L_u(T) \end{array}$$

$r=qp'$. Thus kernel of $p'\subset$kernel of $r=I$, and so kernel of $p'=I$. (Since $\sigma(I)=I$, σ also acts on $L_u(E)/I$ so that $r\sigma=\sigma r$. Since $p'\sigma=\sigma p'$ and p' is onto, $q\sigma=\sigma q$ also.)

We now define the ideal J in $L_u(E)$ as $J=\{x\in[E, M] \mid [x, a]=0 \text{ for all } a\in E\}$. Since $[J, E]=0$ and E generates $L_u(E)$, $[J, L_u(E)]=0$. Also, $\sigma(x)=x$ for all $x\in J$, and so $J\cap E=(0)$.

Let $G=L_u(E)/J$, and j the natural homomorphism of $L_u(E)$ onto G. j is 1-1 on E, and so may be regarded as an inclusion map on E. G is also a Lie algebra with involution σ and $\sigma j=j\sigma$, since $\sigma(J)=J$ in $L_u(E)$. Since $J\subset I$, the projection p' of $L_u(E)$ onto $L_u(T)$ induces a homomorphism p_0 (also commuting with σ) of G onto $L_u(T)$: $p_0j=p$ on $L_u(E)$. $J\supset[M, M]$ since $[x, y, z]=0$ in E whenever two of x, y, z are in M, so $[j(M), j(M)]=0$ in G. The kernel N of p_0 is $j(I)=j(M\oplus[E, M])=j(M)\oplus[j(E), j(M)]$.

Let q be a 1-1 linear map of $T\rightarrow E$ such that $pq(t)=t$ for $t\in T$, then $E=M\oplus q(T)$ as vector space, so $j(E)=j(M)\oplus jq(T)$. The kernel of p_0 may then be written as $N=j(M)\oplus[j(E), j(M)]=j(M)\oplus[jq(T), j(M)]$, since $[j(M), j(M)]=0$.

N may be considered as $L_u(T)$ module by $[t, n]=[jq(t), n]$ (this is independent of the choice of q since $[j(M), j(M)]=0$ and $[[jq(t), j(m)], j(m')]=j([q(t), m, m'])=0$). The induced structure on $j(M)$ as Lie triple system module for T is then isomorphic to M as T-module, under the map $m\rightarrow j(m)$: $[j(m), jq(t), jq(t')]=j([m, t, t'])$. We have to show N is isomorphic to M_s as $L_u(T)$ module.

Since $N=j(M)\oplus[jq(T), j(M)]=j(M)\oplus[T, j(M)]$, there is an $L_u(T)$ homomorphism of N onto M_s taking $j(m)\rightarrow m$, $[j(m), t]\rightarrow[m, t]$. To show this map is 1-1, we suppose that $\sum[j(m_i), t_i]\rightarrow\sum[m_i, t_i]=0$ (the map being known to be 1-1 on $j(M)$). Then $\sum[m_i, t_i, t]=0$ in M for all $t\in T$, and, in E, $\sum[m_i, q(t_i), q(t)]=0$. Thus $\sum[m_i, q(t_i)]\in J$ (since $\sum[[m_i, q(t_i)], m]=0$ automatically for $m\in M$), and, applying j, $\sum[j(m_i), jq(t_i)]=\sum[j(m_i), t_i]=0$ in G.

Finally, it is clear that the given diagram is commutative if j also denotes the inclusion of T into $L_u(T)$. This concludes the proof of the lemma.

To each factor set g of $L_u(T)$ into M_s satisfying $g(\sigma(x), \sigma(y))=\sigma g(x, y)$ (i.e., element of $Z^2_+(L_u(T), M_s)$) we shall assign a factor set $f=\alpha(g)$ as follows: from g we construct an extension G of $L_u(T)$ by M_s as in the last section: $G=M_s\oplus L_u(T)$ as vector space, the involution is $\sigma(n\oplus x)=\sigma(n)\oplus\sigma(x)$, and the multiplication is determined by: $[0\oplus x, 0\oplus y]=g(x, y)\oplus[x, y]$. Thus $E=M\oplus T\subset G$. We then have the exact sequence $0\rightarrow M_s\rightarrow^i G\rightarrow^p L_u(T)\rightarrow 0$, and (by restricting the map i to M, p to $E=T\oplus M$) the exact sequence $0\rightarrow M\rightarrow^i E$

$\to^p T\to 0$, i.e., we also have an extension E of T by M. The multiplication in $E=M\oplus T$ is determined by

$$(3.5)\qquad [[0\oplus t_1, 0\oplus t_2], 0\oplus t_3] = f(t_1, t_2, t_3)\oplus[[t_1, t_2], t_3]$$

where

$$(3.6)\qquad f(t_1, t_2, t_3) = [g(t_1, t_2), t_3] + g([t_1, t_2], t_3).$$

In short, $f=\alpha(g)$ is determined by (3.6). It is then clear that α is a linear map of $Z^2_+(L_u(T), M_s)$ into the space F of factor sets f of T into M.

If the factor set g is trivial, i.e., $\in B^2_+(L_u(T), M_s)$ then, in terms of the extension $G\to^p L_u(T)$, there is a Lie algebra homomorphism q of $L_u(T)$ into G such that $pq(l)=l$ for $l\in L_u(T)$, and $q\sigma=\sigma q$. Thus q maps T into E and so the factor set f is also trivial. Explicitly, if $q(l)=h(l)\oplus l$ where $h\in C^1_+(L_u(T), M_s)$ then f satisfies (3.4) with this function h.

Conversely, suppose $f=\alpha(g)$ and f is trivial. Using the extensions G of $L_u(T)$ by M_s and E of T by M just described, we have a homomorphism of Lie triple systems q of T into E such that $pq(t)=t$ for $t\in T$. Since $L_u(T)$ is the universal Lie algebra of T, and q is a homomorphism of T into the Lie algebra G, q can be extended to a homomorphism q_0 of $L_u(T)$ into G. Since $q_0(T)\subset E$, $q_0([T, T])\subset[E, E]$ and so $q_0\sigma=\sigma q_0$. Finally, $pq_0(t)=t$ for $t\in T$, and so pq_0 is the identity on $L_u(T)$. Thus g is also a trivial factor set.

Finally, the map α is onto F: given $f\in F$, we construct an extension E of T by M with this factor set, i.e., we find a linear inverse q of p such that $f(t_1, t_2, t_3) = [[q(t_1), q(t_2)], q(t_3)]-q([t_1, t_2], t_3])$. By the lemma, we can construct a corresponding extension G of $L_u(T)$ by M_s such that the projection p_0 of G onto $L_u(T)$ agrees with p on E. Then q can be extended to a linear inverse q_0 of p_0, mapping T into E, $[T, T]$ into $[E, E]$. If $g(x, y)=[q_0(x), q_0(y)]-q_0([x, y])$ then $f(t_1, t_2, t_3)=[g(t_1, t_2), t_3]+g([t_1, t_2], t_3)$ so $f=\alpha(g)$. Thus α is an isomorphism of $H^2_+(L_u(T), M_s)$ onto the group of factor sets of T into M modulo trivial factor sets.

For the rest of this section we shall specialize to Φ of characteristic zero and finite-dimensional T and M. The radical of T can then be defined (see [9]) as the maximal solvable ideal (an ideal I being a subspace such that $[I, T, T]\subset I$, and solvable meaning that the series $I^{(0)}=I, I^{(n+1)}=[T, I^{(n)}, I^{(n)}]$ terminates with zero). If the radical is zero, T is called semi-simple, and this condition is exactly equivalent to $L_u(T)$ being semi-simple. If T is semi-simple, it is a direct sum of simple ideals T_i and $L_u(T)$ is the direct sum of the $L_u(T_i)$. If T is simple, then either $L_u(T)$ is a simple Lie algebra and T is *not* isomorphic to a Lie algebra considered as Lie triple system, or else T is isomorphic to a simple Lie algebra L considered as Lie triple system and $L_u(T)=L\oplus L$ with involution $\sigma(a\oplus b)=b\oplus a$: in other words, $L_u(T)$ is simple when considered as Lie algebra with involution, i.e., it has no proper self-adjoint ideals.

THEOREM 3.3. *Let T be a finite-dimensional semi-simple Lie triple system over a field of characteristic zero, M a finite-dimensional module. Then* $H^1(T, M) = 0 = H^2(T, M)$.

Proof. $L_u(T)$ is finite-dimensional and semi-simple and M_s is finite-dimensional, thus $H^1(L_u(T), M_s) = 0 = H^2(L_u(T), M_s)$. But $H^n_+(L_u(T), M_s) = H^n(T, M)$ is a direct summand of $H^n(L_u(T), M_s)$, and so is also zero for $n = 1, 2$.

COROLLARY. *Under the same hypotheses as in Theorem* 3.3, *every derivation of T into M is inner, and every factor set of T into M is trivial.*

As shown in [9], the statement on factor sets for semi-simple T can be used to prove the Levi decomposition for any finite dimensional Lie triple system E: $E = T + R$, T semi-simple, R the radical. The Malcev-Harish-Chandra uniqueness theorem for the Levi decomposition is easy to prove: it states that, if $E = T_1 + R = T_2 + R$, T_i semi-simple, then T_1 is carried onto T_2 by an automorphism of E of the form exp(Ad z), $z \in [E, R] \subset$ radical of $L_u(E)$, i.e., the automorphism is

$$x \to x + \sum_i [u_i, v_i, x] + \frac{1}{2} \sum_{i,j} [u_i, v_i, ([u_j, v_j, x]) + \cdots \text{ for } u_i \in E, v_i \in R.$$

Such automorphisms form a group since $[E, R]$ is a subalgebra of $L_u(E)$.

PROPOSITION 3.2. *If T is finite-dimensional semi-simple over a field of characteristic zero, then every finite-dimensional module M is completely reducible.*

Proof. Let M' be a submodule of M; then M'_s can be considered as a submodule of M_s and is mapped into itself by σ, so that the factor module $P = M_s/M'_s$ can be also considered as module with involution for $L_u(T)$. The sequence of K-modules $0 \to M'_s \to^{i_0} M_s \to P \to 0$ splits, since

$$\mathrm{Ext}^1_K(P, M_s) \simeq H^1_+(L_u(T), \mathrm{Hom}_\Phi(P, M_s)) = 0,$$

and so there is a homomorphism of M_s onto M'_s inverse to i_0 and commuting with σ. This homomorphism maps M onto M', so M' has a complement in M.

We can say something about the third cohomology group in case the module $M = \Phi$ (with the elements of T being zero-operators) so that $M_s = \Phi$ also:

THEOREM 3.4. *Let T be simple and finite-dimensional over an algebraically closed field Φ of characteristic zero. Then $H^3(T, \Phi)$ is either zero or one-dimensional, according as $L_u(T)$ is simple or not.*

Proof. If $L_u(T)$ is simple, then $H^3(L_u(T), \Phi)$ is one-dimensional and spanned by the class of the cocycle $f(x, y, z) = B([x, y], z)$ where $B(x, y)$ is the invariant symmetric bilinear form tr(Ad x Ad y) on $L_u(T)$ (see [8]). Since σ is an automorphism, $\mathrm{Ad}(\sigma x) = \sigma \cdot \mathrm{Ad}\, x \cdot \sigma^{-1}$ so $B(\sigma x, \sigma y) = B(x, y)$ and

$f(\sigma x, \sigma y, \sigma z) = f(x, y, z) = -\sigma f(x, y, z)$, so that $f \in Z^3_-(L_u(T), \Phi)$ and $H^3_+(L_u(T), \Phi) = 0$.

Now suppose $L_u(T)$ is not simple, and so is the direct sum $L_1 \oplus L_2$ of simple ideals mapped onto one another by σ. Denote $L_u(T)$ by L. Then the exterior algebra $\Lambda(L) \simeq \Lambda(L_1) \otimes \Lambda(L_2)$ (tensor product of anti-commutative graded algebras), and so the dual space $\Lambda^*(L) \simeq \Lambda^*(L_1) \otimes \Lambda^*(L_2)$. The cohomology algebra $H(L) \simeq H(L_1) \otimes H(L_2)$. Since the L_i are simple, $H^n(L_i) = H^n(L_i, \Phi)$ is zero-dimensional for $n = 1, 2$, and one-dimensional for $n = 0, 3$. Thus $H^3(L) = H^3(L, \Phi)$ is two-dimensional, spanned by the cohomology classes of $f_1 \otimes 1^* + 1^* \otimes f_2$ and $f_1 \otimes 1^* - 1^* \otimes f_2$ where the f_i are the 3-cocycles of L_i described above and chosen so that $f_1(\lambda) = f_2(\sigma\lambda)$ for $\lambda \in \Lambda^3(L_1)$. Thus for $\lambda \in \Lambda^3(L_1)$, $\sigma\lambda \in \Lambda^3(L_2)$, and $(f_1 \otimes 1^* + 1^* \otimes f_2)(\sigma\lambda) = f_2(\sigma\lambda) = f_1(\lambda) = (f_1 \otimes 1^* + 1^* \otimes f_1)(\lambda)$ so $f_1 \otimes 1^* + 1^* \otimes f_2 \in Z^3_-$, and similarly $f_1 \otimes 1^* - 1^* \otimes f_2 \in Z^3_+$. Thus $H^3(T, \Phi) = H^3_+(L_u(T), \Phi)$ is one-dimensional.

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NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS